CECS-PHY-04/13

Vacuum Energy in Odd-Dimensional AdS Gravity

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Abstract. A background-independent, Lorentz-covariant approach to compute conserved charges in odd-dimensional AdS gravity, alternative to the standard counterterms method, is presented. A set of boundary conditions on the asymptotic extrinsic and Lorentz curvature, rather than a Dirichlet boundary condition on the metric is used. With a given prescription of the boundary term, a well-defined action principle in any odd dimension is obtained. The same boundary term regularizes the Euclidean action and gives the correct black hole thermodynamics. The conserved charges are obtained from the asymptotic symmetries through Noether theorem without reference to any background. For topological AdS black holes the vacuum energy matches the expression conjectured by Emparan, Johnson and Myers [1] for all odd dimensions.

The development of background-independent methods to compute conserved charges for gravity has attracted considerable attention in the recent literature. A clear advantage of these methods over the time-honored Hamiltonian approach [2] and other background-substraction procedures [3], is that they do not require to specify a reference background configuration.

Inspired in the AdS/CFT conjecture [4,5], the counterterms method proposes a regularization scheme that preserves general covariance [6]. Adding invariants of the boundary metric to the bulk action (supplemented by the Gibbons-Hawking term [7]) a finite stress tensor for AdS spacetimes is obtained [1,8]. Even though this approach correctly gives the conserved charges for a number of solutions, its main drawback is the proliferation of possible terms as the complexity of the solution and the spacetime dimension increase (see, e.g., [9,10]), and the full series for any dimension remains unknown.

On the other hand, the use of boundary conditions in AdS gravity different from Dirichlet's on the metric has proved to be a good alternative to produce a finite action principle [11–13]. In even dimensions, regularized conserved charges are constructed for Einstein-Hilbert and other theories with higher powers in the curvature, imposing a boundary condition on the asymptotic curvature. In this case, the action is supplemented by the Euler term carrying different weight factors depending on the theory.

The same approach cannot be applied to odd-dimensional AdS gravity, because there are no topological invariants of the Euler class in $d = 2n + 1$, a fact that makes gravity in even and odd dimensions radically different. Obviously, it is always possible to add arbitrary boundary terms to the bulk action, but unless there is a clear guideline to construct them, this procedure faces the same difficulties as the counterterms method.

New insight on the above problem was gained by the introduction of a boundary condition for Chern-Simons-AdS gravity [14]. In the spirit of the even-dimensional case, where a single (boundary) term yields a finite action principle for a family of inequivalent gravity theories, one may expect that the same boundary term for Chern-Simons gravity will also set a well-defined action principle for General Relativity with $\Lambda < 0$ (indeed, both theories coincide in three dimensions, and so do their boundary terms).

In this article, we propose an alternative mechanism to regularize the action and the conserved charges in odd-dimensional Einstein-Hilbert-AdS gravity. A set of boundary conditions on the asymptotic extrinsic and Lorentz curvatures singles out the correct boundary term in any odd dimension that solves at once the following problems: (i) the variation of the action vanishes *on-shell*, (ii) background-independent conserved charges, which give the correct mass without need for substractions or *ad hoc* regularizations (iii) finite Euclidean action, and the right entropy and thermodynamics for black hole solutions.

Action principle. For any odd dimension, $d = 2n+1$, the action for General Relativity with negative cosmological constant is [15]

$$I_g = \kappa \int\limits_{\mathcal{M}} \hat{\epsilon}_{A_1 \ldots A_d} (\hat{R}^{A_1 A_2} e^{A_3} \ldots e^{A_d} + \frac{d-2}{l^2 d} e^{A_1} \ldots e^{A_d}) + \kappa \alpha_{2n} \int\limits_{\partial\mathcal{M}} B_{2n} \qquad (1)$$

where $e^A = e^A_\mu dx^\mu$ represents the vielbein and $\hat{R}^{AB} = \hat{R}^{AB}_{\mu\nu} dx^\mu \wedge dx^\nu$ is the 2-form Lorentz curvature constructed up from the spin connection $\omega^{AB} = \omega^{AB}_\mu dx^\mu$ as $\hat{R}^{AB} = d\omega^{AB} + \omega^A_C \omega^{CB}$. The wedge product $\wedge$ between the differential forms is understood.

The gravitational bulk action has been supplemented by an appropriate boundary term B_{2n}, whose explicit

form relies on the boundary condition chosen to have a well-defined action principle. The field equations for (1) are obtained varying with respect to the dynamical fields, e^A and ω^{AB} yields

$$\delta I_G = \kappa \int\limits_M \varepsilon_A \delta e^A + \varepsilon_{AB} \delta \omega^{AB} + d\Theta, \tag{2}$$

where ε_A is the Einstein tensor,

$$\varepsilon_A = \kappa \hat{\epsilon}_{AA_2 \ldots A_d} \left(\hat{R}^{A_2 A_3} + \frac{1}{l^2} e^{A_2} e^{A_3} \right) e^{A_4} ... e^{A_d}. \tag{3}$$

Assuming that the vielbein is invertible, the equation $\varepsilon_{AB} = 0$ simply implies that the torsion must vanish.

The surface term Θ contains two contributions, one coming from integration by parts the bulk action, and the other from original boundary term,

$$\Theta = \kappa \left(\hat{\epsilon}_{A_1 A_2 A_3 \ldots A_d} \delta \omega^{A_1 A_2} e^{A_3} \ldots e^{A_d} + \alpha_{2n} \delta B_{2n} \right). \tag{4}$$

Boundary conditions. In what follows we will consider a radial foliation of the spacetime, such that the line element is written as

$$ds^2 = N^2(r) dr^2 + h_{ij}(r,x) dx^i dx^j \tag{5}$$

and we will adapt the vielbein to the boundary geometry, $e^1 = N dr$ and $e^a = e^a_i dx^i$, where we have split the tangent space indices as $A = \{1, a\}$ and the spacetime ones as $\mu = \{r, i\}$.

The vanishing of torsion means that the spin connection is completely determined by the vielbein as $\omega^{AB}_\mu = -e^{B\nu} \nabla_\mu e^A_\nu$, where ∇_μ is the covariant derivative in the Christoffel symbol. In particular, the components ω^{1a} are related to the vielbein at the boundary by

$$\omega^{1a} = -K^j_i e^a_j dx^i = -K^a \tag{6}$$

where K_{ij} is the extrinsic curvature that in the adapted coordinates frame (5) is $K_{ij} = -\frac{1}{2N} h'_{ij}$ (a prime is used to denote radial derivative).

The boundary term B_{2n} must be expressible as a function of the vielbein, the spin connection and the Lorentz curvature at the boundary. Its expression should match the standard tensorial formulation, where the boundary terms are written as local functions of the boundary metric h_{ij}, the extrinsic curvature K_{ij} and intrinsic curvature R^{kl}_{ij} of the boundary metric. Both languages are naturally related if the rotational symmetry of the fields is fixed by choosing a preferred frame at the boundary.

The second fundamental form (SFF) is defined as the difference of two spin connections at the boundary,

$$\theta^{AB} = \omega^{AB} - \bar{\omega}^{AB}, \tag{7}$$

where ω^{AB} is the dynamical field and $\bar{\omega}^{AB}$ is a fixed reference at the boundary. Lorentz covariance is recovered if $\bar{\omega}^{AB}$ is assumed to transform under $SO(d-1,1)$ in the same way as ω^{AB}. That does not occur if the generators of Lorentz rotations are represented as functions of the physical fields in phase space. In that case, $\bar{\omega}^{AB}$ would not transform, thus breaking Lorentz covariance at the boundary.

Inspired by the matching conditions that single out the boundary term in the Euler theorem [16], we take a connection $\bar{\omega}^{AB}$ such that at ∂M it satisfies

$$\bar{\omega}^{1a} = 0, \;\; \bar{\omega}^{ab} = \omega^{ab}. \tag{8}$$

This corresponds to the case where $\bar{\omega}^{AB}$ represents the connection in a cobordant manifold that is locally a product space of the boundary ∂M and a trivial extension along the normal direction. Then the SFF has only 'normal' components,

$$\theta^{1a} = -K^a_i dx^i, \;\; \theta^{ab} = 0. \tag{9}$$

As the boundary is located at fixed r, it does not admit form components containing dr. Thus, in ∂M, the Gauss-Coddazzi decomposition for the Riemann two-form $\hat{R}^{AB}$ is given by

$$\hat{R}^{1a} = D_i(\omega) \theta^{1a}_j dx^i \wedge dx^j, \tag{10}$$

$$\hat{R}^{ab} = \left(R^{ab}_{ij} + \theta^a_{i1} \theta^{1b}_j \right) dx^i \wedge dx^j. \tag{11}$$

It must be stressed that the connection $\bar{\omega}^{ab}$ is only defined at the boundary ∂M, where it matches the dynamical ω^{ab}. This approach does not require to specify the geometry of a background (vacuum solution) throughout the spacetime M, as in the background substraction aproach.

Equipped with these ingredients, we shall prove below that the same Lorentz-covariant boundary term considered in [14] makes the action for GR stationary under arbitrary variations of the fields:

$$B_{2n} = -n \int\limits_0^1 dt \int\limits_0^t ds \hat{\epsilon} \theta e \left(R + t^2 \theta\theta + s^2 \frac{ee}{l^2} \right)^{n-1}. \tag{12}$$

Here the Lorentz indices have been omitted to simplify the notation $\hat{\epsilon} =: \hat{\epsilon}_{A_1 A_2 A_3 \ldots A_d}$ and $\theta\theta =: (\theta\theta)^{AB} = \theta^A_C \theta^{CB}$. The integration over the continuous parameters t, s in the interval $[0,1]$ gives the relative coefficients in the series of terms in B_{2n}. It will now be shown that under the appropriate boundary conditions the surface term Θ vanishes identically.

The total surface term (4) can be shown to be [17]

$$\Theta = \kappa \epsilon \delta K e^{2n-1} - \kappa \alpha_{2n} n \int\limits_0^1 dt \epsilon \delta K e \left(R - KK + t^2 \frac{ee}{l^2} \right)^{n-1}$$

$$+ \kappa \alpha_{2n} n \int\limits_0^1 dt t \epsilon (\delta K e - K \delta e) \left(R - t^2 KK + t^2 \frac{ee}{l^2} \right)^{n-1} \tag{13}$$

where ϵ is the Levi-Civita tensor of the boundary, $\epsilon_{a_1...a_{2n}} = -\hat{\epsilon}_{1a_1...a_{2n}}$.

The action is to be varied under the asymptotic conditions

$$K^i_j = \delta^i_j \tag{14}$$

$$\delta K_{ij} = 0 \tag{15}$$

for the extrinsic curvature which, in view of (6), means $\delta K^a = \delta e^a$. Thus, the third term in (13) vanishes identically.

By definition, K_{ij} is the Lie derivative along a normal to the boundary, $K_{ij} = \mathcal{L}_n h_{ij}$. Then the boundary condition (14) means that the boundary admits a conformal Killing vector. In this sense, this boundary condition is *holographic* and was first introduced in the context of odd-dimensional Chern-Simons-AdS gravity [14]. An embedded manifold that satisfies this condition is also known as *totally umbilical* [18].

Additionally, spacetime is assumed to be asymptotically locally anti-de Sitter (**ALAdS**),

$$\hat{R}^{ab} = R^{ab} - K^a K^b = -\frac{1}{l^2} e^a e^b. \tag{16}$$

Then, Θ vanishes if the weight factor α_{2n} is fixed as

$$\alpha_{2n} = \frac{l^{2(n-1)}}{n}\left[\int_0^1 dt(t^2-1)^{n-1}\right]^{-1} = (-l^2)^{n-1}\frac{(2n-1)!!}{n!2^{n-1}}, \tag{17}$$

and the action has indeed an extremum on-shell.

It is worth mentioning that the ALAdS condition reflects a local property at the boundary, but it does not restrict the global topology of the spacetime manifold. In fact, there is a wide class of solutions that satisfy this condition, including black holes, black strings, Kerr-AdS, Taub-NUT/Bolt-AdS, etc.

Conserved Charges. According to Noether's theorem, a conserved current associated to the invariance under diffeomorphisms of a d-form Lagrangian L, is given by [19,20]

$$*J = -\Theta(\varphi, \delta\varphi) - I_\xi L \tag{18}$$

where Θ is the boundary term in (2) with the variations of the fields given by their Lie derivatives, $\delta\varphi = -\mathcal{L}_\xi \varphi$, and I_ξ is the contraction operator [21].

In this way, the current can be written as an exact form, $*J = dQ(\xi)$. As long as the fields are smooth in the asymptotic region, the conserved charge takes the form of a surface integral

$$Q(\xi) = \kappa \int_{\partial\Sigma} \epsilon I_\xi K \left(e^{2n-1} - \alpha_{2n} n \int_0^1 dte(\hat{R} + t^2\frac{ee}{l^2})^{n-1} \right)$$
$$+\alpha_{2n} n \int_0^1 dtt\epsilon \left(I_\xi K e + K I_\xi e\right)\left(R - t^2 KK + t^2\frac{ee}{l^2}\right)^{n-1} \tag{19}$$

This expression defines a useful conserved charge if the parameter ξ is an asymptotic Killing vector. In spite of the freedom to add an arbitrary closed form to the current, once a well-defined action principle is established, Noether's theorem leads to the correct conserved charges [11–14]. Next, a concrete example is shown.

Topological Static Black Holes. Formula Eq.(19) can be evaluated for static (topological) black holes whose line element is given by

$$ds^2 = \Delta(r)^2 dt^2 + \frac{dr^2}{\Delta(r)^2} + r^2 d\Sigma^2_\gamma \tag{20}$$

with $\Delta^2 = \gamma - \frac{2G\mu}{r^{2(n-1)}} + \frac{r^2}{l^2}$ and $d\Sigma^2_\gamma$ is the line element of the $(d-2)-$dimensional transverse section Σ_γ of constant curvature $\gamma = \pm 1, 0$, and volume Σ_{d-2}.

For $\xi = \partial_t$, the charge has two pieces, $Q(\partial_t) = E + E_0$, coming from the first and the second line of (19). They are the mass and the AdS *zero point (vacuum) energy*, respectively. The mass is

$$E = \frac{\Sigma_{d-2}}{\Omega_{d-2}}\mu, \tag{21}$$

in agreement with the Hamiltonian [2] and also the background-independent methods [1]. The zero point energy is given by

$$E_0 = (-1)^n \frac{(2n-3)!!}{n!2^n}\frac{\Sigma_{d-2}}{\Omega_{d-2}G}\gamma^n l^{2(n-1)}. \tag{22}$$

When expressed in units such that the entropy is $S = Area/4G_N$ [15] this vacuum energy takes the form

$$E_0 = \frac{\Sigma_{d-2}}{8\pi G_N}\left((-\gamma)^n \frac{(2n-1)!!^2}{(2n)!}\right) l^{2(n-1)}, \tag{23}$$

confirming the expression for all odd dimensions proposed by Emparan, Johnson and Myers [1].

Black Hole Thermodynamics. Static black hole solutions (20) possess an event horizon at r_+ ($\Delta^2(r_+) = 0$) and whose topology is given by the transverse section Σ. The black hole temperature T is defined by the requirement that, in the Euclidean sector, the solution be smooth at the horizon. This fixes the period of the Euclidean time as

$$\beta = T^{-1} = \frac{1}{4\pi}\left(\left.\frac{d\Delta^2}{dr}\right|_{r_+}\right)^{-1} = \frac{2\pi}{\left[n\frac{r_+}{l^2} + \gamma\frac{(n-1)}{r_+}\right]}. \tag{24}$$

In the canonical ensemble, the Euclidean action I_E is given by the free energy, $I_E = -\beta F = S - \beta\widetilde{E}$ that defines the *energy* and the entropy of a black hole for a fixed surface gravity (temperature). The Wick-rotated version of the bulk action (1), evaluated on-shell is

$$I_E^{bulk} = -\frac{\beta}{(d-2)G}\frac{\Sigma_{d-2}}{\Omega_{d-2}}\left.\left(\frac{r^{2n}}{l^2}\right)\right|_{r+}^{\infty}, \tag{25}$$

and the Euclidean boundary term

$$\kappa\alpha_{2n}B_{2n}^{E}=-\frac{\beta}{(d-2)G}\frac{\Sigma_{d-2}}{\Omega_{d-2}}\left[\mu G-\frac{\alpha_{2n}}{2}\gamma^{n}-\left(\frac{r^{2n}}{l^{2}}\right)\Big|^{\infty}\right], \tag{26}$$

exactly cancels the divergence at radial infinity. Then, using the fact that $\mu=\frac{r_{+}^{d-3}}{2G}\left(\gamma+\frac{r_{+}^{2}}{l^{2}}\right)$, the Euclidean action is finite and given by

$$I_{E}=\frac{\beta}{2(d-2)G}\frac{\Sigma_{d-2}}{\Omega_{d-2}}\left[r_{+}^{d-3}(r_{+}^{2}-\gamma)+\alpha_{2n}\gamma^{n}\right]. \tag{27}$$

The energy appears again shifted by a constant with respect to the Hamiltonian mass,

$$\widetilde{E}=-\frac{\partial I_{E}}{\partial\beta}=-\frac{\partial I_{E}/\partial r_{+}}{\partial r_{+}/\partial\beta}=\frac{\Sigma_{d-2}}{\Omega_{d-2}}\mu+E_{0}, \tag{28}$$

where E_0 turns out to be the same quantity found as the vacuum energy in Eq.(23). Finally, the standard result for the black hole entropy is recovered,

$$S=\frac{2\pi r_{+}^{d-2}}{(2n-1)G}\frac{\Sigma_{d-2}}{\Omega_{d-2}}=\frac{r_{+}^{d-2}\Sigma_{d-2}}{4G_{N}}=\frac{Area}{4G_{N}}. \tag{29}$$

Conclusions. A Lorentz-covariant boundary term B_{2n} for the GR action with negative cosmological constant in $2n+1$ dimensions is introduced. This renders the action stationary on shell under arbitrary variations subject to a specific boundary conditions. This "counterterm" is also shown to regularize the action as well as the conserved charges associated to asymptotic symmetries for Schwarzschild-AdS black holes and their topological extensions.

The geometry is asymptotically locally AdS, and the fields at the boundary ∂M are such that the spin connection of the boundary is prescribed and ∂M admits a conformal Killing vector in the radial direction (∂M is totally umbilical). It is worth noticing that the converse argument is also true: had we started with the surface term coming from the variation of the bulk action in Eq.(4), we would have been able to integrate out the boundary term B_{2n} from the variations of the fields using the boundary condition (14) and the asymptotic property (16).

It would be interesting to explore the implications of this method in the context of AdS/CFT correspondence and, in particular, the holographic reconstruction in AdS spacetimes. Contrary to the formalism developed by Henningson and Skenderis [6], where they solve the asymptotic Einstein equations for the Dirichlet problem (a given boundary metric), this time the initial data is a given value of the extrinsic curvature at the boundary.

Acknowledgments

We wish to thank M. Bañados, G. Kofinas, O. Mišković and S. Theisen for helpful discussions. PM and RO are grateful to Centro de Estudios Científicos, for the hospitality during the completion of this work. RO and JZ thank Prof. S. Theisen for hospitality at AEI, Golm. This work was partially funded by the grants 1010450, 1010449, 1020629, 1040921, 3030029 and 7010450 from FONDECYT. Institutional support to Centro de Estudios Científicos (CECS) from Empresas CMPC is acknowledged. CECS is a Millennium Science Institute and is funded in part by grants from Fundación Andes and the Tinker Foundation.

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